

INTERLACEMENT OF DOUBLE CURVES OF IMMERSED SPHERES

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ABSTRACT. We characterize those unions of embedded disjoint circles in the sphere S^2 which can be the multiple point set of a generic immersion of S^2 into \mathbb{R}^3 in terms of the interlacement of the given circles. Our result is the one higher dimensional analogue of Rosenstiehl's characterization of words being Gauss codes of self-crossing plane curves. Our proof uses a result of Lippner [Li04] and we further generalize the ideas of Fraysseix and Ossona de Mendez [FO99], which leads us to directed interlacement graphs of paired trees and their local complementation.

1. INTRODUCTION

Our goal is to understand the multiple point sets of “nicely” mapped spheres into 3-dimensional Euclidean space. As explained in this section below, our task in this paper will be to manipulate certain interlaced systems of smoothly embedded circles in S^2 . Although our main results are topological-combinatorial (see Definition 1.1, Remark 1.2 (1) and Theorem 2.1), we take a combinatorial-algebraic viewpoint during the proof in order to make some analogies with [FO99] transparent. We hope readers interested in either combinatory or topology will benefit from this approach. To get motivated, we list two problems: first a topological and then a combinatorial one.

- (1) Take a 2-knot, i.e. a smooth embedding $e: S^2 \rightarrow \mathbb{R}^4$, and take the map $\pi \circ e$, where $\pi: \mathbb{R}^4 \rightarrow \mathbb{R}^3$ is the natural projection. It is well-known that $\pi \circ e$ is a stable map after a small perturbation of e , moreover $\pi \circ e$ becomes a generic immersion of S^2 into \mathbb{R}^3 after an appropriate isotopy of e . The multiple points of this generic immersion form an immersed curve in S^2 and one can ask what is the criterion for immersed curves in S^2 to come up in this way. This question is the one dimension higher analogue of the problem of chord diagrams representing

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knot diagrams [FO99, LM76, RR78]. In the 2-knot case, restricting ourselves to curves in S^2 which are disjoint unions of embedded circles is related to asking about ribbon 2-knots and their possible double decker sets, see, for example [CKS04]. So our problem to study is whether a *virtual* ribbon 2-knot (i.e. a disjoint union of embedded circle pairs in S^2) is *classical* (i.e. comes from a 2-knot diagram).

- (2) Take a circle graph (the intersection graph of chords of the unit circle in the plane), it is an undirected graph. It is well-known that local complementation at its vertices corresponds to switching at the chords (for details, see, for example [Bou94, FO99] and also [De36]). One can ask what is the corresponding class of *directed* graphs, i.e. whether performing a directed version of local complementation at a vertex of a directed graph G is related to some switching-like operation on an object such that G can be interpreted as the interlacement graph of this object. In the present paper we show that the directed interlacement graph of a paired tree (defined appropriately) has this property.

Paired trees and realization. A disjoint union of smoothly embedded circles

$$C_1, \dots, C_{n-1}$$

in S^2 determines a tree, where the vertices v_1, \dots, v_n of the tree correspond to the connected components A_1, \dots, A_n of $S^2 - (C_1 \cup \dots \cup C_{n-1})$ and the edges e_1, \dots, e_{n-1} of the tree represent the circles themselves along which A_1, \dots, A_n are attached together. In a similar fashion any tree gives us a disjoint union of embedded circles in S^2 , which is unique up to self-diffeomorphisms of S^2 . We consider the embedded circles up to such self-diffeomorphisms of S^2 and hence we have a bijection between trees and disjoint unions of embedded circles. We say that a tree is *paired* if it has an even number of edges, which are arranged into pairs. A paired tree or the union of the corresponding paired circles is called *realizable* if there is a generic immersion of S^2 into \mathbb{R}^3 whose multiple point set is equal to the paired circles (so the generic immersion has no triple points and we assume that two circles in S^2 form a pair if and only if their images are equal as sets under the immersion, for example, see Figure 1).

In the present paper, we study when paired trees (or equivalently disjoint unions of pairs of embedded circles in S^2) are realizable. We expect conditions which are phrased in terms of the relative positioning of the circle pairs in S^2 . In the simplest way this

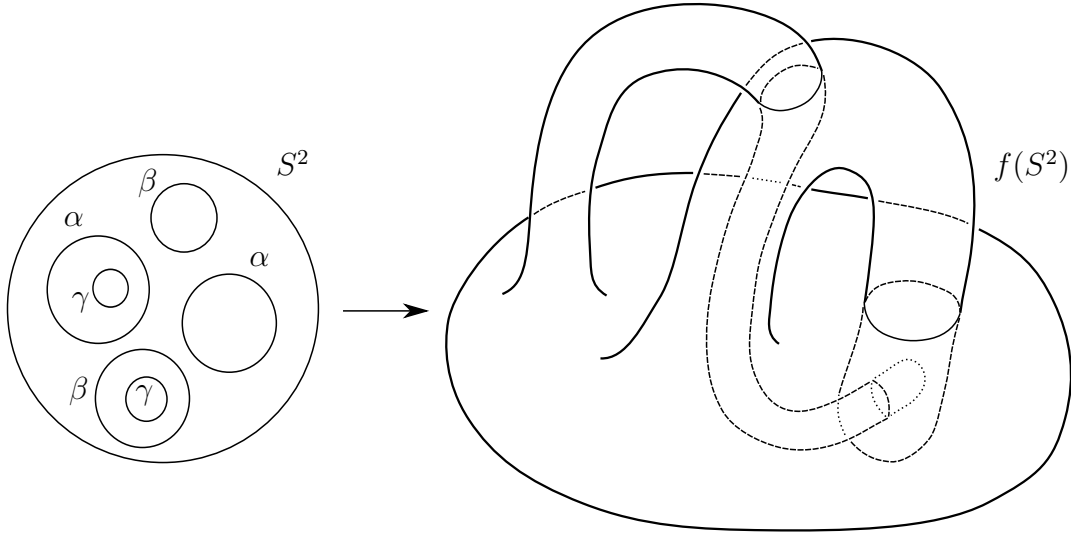


FIGURE 1. A generic immersion $f: S^2 \rightarrow \mathbb{R}^3$ without triple points. On the left hand side we can see the disjoint union of the embedded circle pairs α , β and γ in S^2 , which are the double curves of f .

would lead to a binary relation on the set of the given circle pairs measuring how they are interlaced.

Interlacement of circle pairs. Let $k \geq 1$ be fixed and $C \subset S^2$, $C = C_1 \cup \dots \cup C_{2k}$ be a disjoint union of $2k$ embedded circles. We suppose that the components of C are arranged into the pairs $\mathbf{c}_j = \{C_j, C_{j+k}\}$, $1 \leq j \leq k$. We define a binary relation \Subset on the set of circle pairs $\{\mathbf{c}_1, \dots, \mathbf{c}_k\}$ as follows.

Definition 1.1 (Interlacement). Let $1 \leq i \leq k$ and let $A_i \subset S^2$ denote the annulus bounded by $C_i \cup C_{i+k}$. Then for a $1 \leq j \leq k$ the relation $\mathbf{c}_j \Subset \mathbf{c}_i$ holds if and only if $i \neq j$ and there is a smooth generic embedded arc α in A_i connecting C_i and C_{i+k} such that the number of intersection points $\alpha \cap (C_j \cup C_{j+k})$ is odd.

Remark 1.2.

- (1) $\mathbf{c}_j \Subset \mathbf{c}_i$ if and only if exactly one of C_j, C_{j+k} represents the generator of the homology group $H_1(A_i; \mathbb{Z}_2)$.
- (2) The relation \Subset is irreflexive and not necessarily transitive or symmetric.

The simplest way to visualize and work with such a relation is to form the *interlacement graph* of the circle pairs.

Definition 1.3 (Interlacement graph). The *interlacement graph* G of C is defined as follows. The vertices v_1, \dots, v_k of G correspond to the circle pairs $\mathbf{c}_1, \dots, \mathbf{c}_k$, respectively, and a directed edge goes from v_i to v_j if and only if $\mathbf{c}_i \subseteq \mathbf{c}_j$.

It is important to note that unlike many similar interlacement relations (coming from circle graphs for example), our interlacement relation is not necessarily symmetric and our interlacement graph is a directed graph¹.

For better understanding, we give the corresponding relation on paired trees as well. Since disjoint unions of circles embedded in S^2 and arranged into pairs are in bijection with paired trees, the relation \subseteq gives a corresponding relation between the pairs of edges of a paired tree, which we denote by \sqsubset . In a paired tree T let $\{a_i, b_i\}, \{a_j, b_j\}$ be two pairs of edges. Then

$$\{a_j, b_j\} \sqsubset \{a_i, b_i\}$$

if and only if there is a path in T connecting a_i to b_i and containing exactly one of a_j, b_j , see Figure 2.

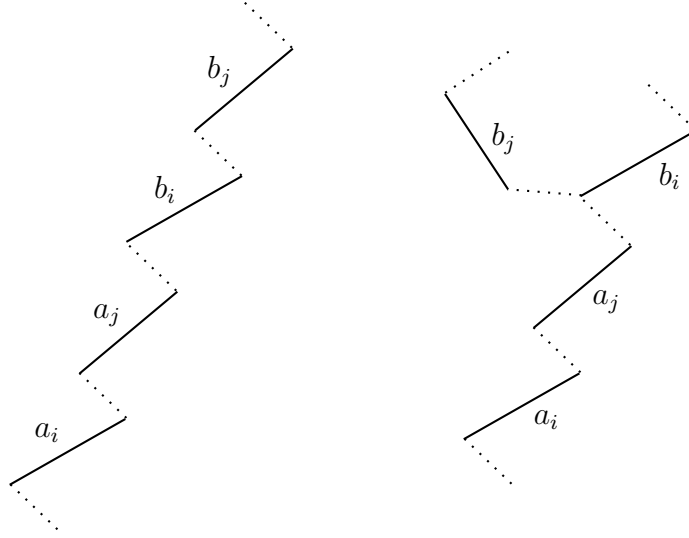


FIGURE 2. The two possible cases of $\{a_j, b_j\} \sqsubset \{a_i, b_i\}$. Note that in the picture on the left hand side also $\{a_i, b_i\} \sqsubset \{a_j, b_j\}$, cf. the two possible cases of linked pairs in [Li04, Figure 6]. Although it may be a little confusing, our motivation to use the term ‘interlacement’ is to express the fact that the relations \subseteq and \sqsubset are typically not symmetric while the ‘linking’ in [Li04] is always symmetric.

¹In [FO99] the authors call two intersecting chords of a circle *interlacement* and it is a symmetric relation.

Recall that the *linking graph* G of a paired tree T is an undirected graph defined in [Li04] whose vertices correspond to the pairs of edges of the paired tree and two vertices of G are connected by an edge if and only if the two corresponding pairs of edges $\{a_1, b_1\}$ and $\{a_2, b_2\}$ of the tree T are linked, i.e. there exists $i, j \in \{1, 2\}$, $i \neq j$, such that the unique path from the edge a_i to the edge b_i contains a_j or b_j but not both, see [Li04, Figure 6]. Note that our interlacement graph determines the linking graph of [Li04] but the converse is not true. In fact two circle pairs \mathbf{c}_j and \mathbf{c}_i are linked if and only if $\mathbf{c}_j \subseteq \mathbf{c}_i$ or $\mathbf{c}_i \subseteq \mathbf{c}_j$.

Local complementation. A version of (directed) local complementation of directed graphs will be used later in the paper, which we define here (cf. local complementation in [Bou87]). Let G be a directed graph and let p be a vertex of G . The operation called local complementation at the vertex p creates a new graph H from the graph G as follows. We will say two vertices of a directed graph are connected if there is an edge (directed arbitrarily) between them. Denote by $NG(u)$ the subgraph of G whose vertices are connected (or identical) to the vertex u .

First we add a new vertex q to the graph G , so H has one more vertex than G . The vertices of H different from q are in natural bijection with the vertices of G and we denote them with the same symbol. Then we have

- (i) $NH(p) = NG(p)$,
- (ii) $NH(q) = NG(p)$,
- (iii) for any vertex u of G we have $NH(u) = NG(u)$ if u is not connected by any edge to p in G and
- (iv) for any two vertices u, v in H not equal to p or q if u has an edge going into p and p has an edge going into v , then u has an edge going into v in H if and only if it has no such edge in G .

For an example, see Figure 3. Note that in the new graph H there is no edge between the vertices p and q . Local complementation has a close relationship with basis change in \mathbb{Z}_2 -vector spaces and matrices, see later.

The double switch operation. A characterization of realizable paired trees was given already by [Li04]. In order to characterize realizable paired trees, [Li04] introduced an operation called double switch (c.f. the operation D-switch in [FO99]). The double switch of a paired tree T along a pair of edges $\{a, b\}$ yields another paired tree T' with one more pair of edges as follows. Take the smallest path P in T containing a and b ,

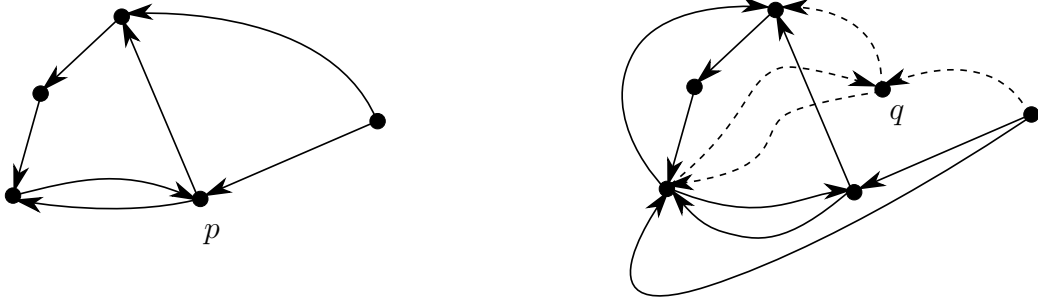


FIGURE 3. Local complementation at the vertex p of the graph on the left hand side transforms it into the graph on the right hand side. The edges starting from and going to the new vertex q are drawn with different style.

denote the ending vertices of P by α and β . Then cut the tree T at α and β and glue the (possibly empty) components not containing P back at the opposite places: the component which was connected to α now is connected to β and vice versa. Also put a new vertex u in the interior of a dividing it into a' , a'' and a new vertex v in the interior of b dividing it into b' , b'' , this gives the new tree T' . Obviously the edges of $T' - a' - a'' - b' - b''$ ² are naturally identified with the edges of $T - a - b$. Then define the pairing of T' to coincide with the pairing of T on the edges of $T' - a' - a'' - b' - b''$ and to be $\{a', b'\}$ and $\{a'', b''\}$ where we suppose that going along a path in T' containing these edges gives the order a', a'', b'', b' , for example, see Figure 4.

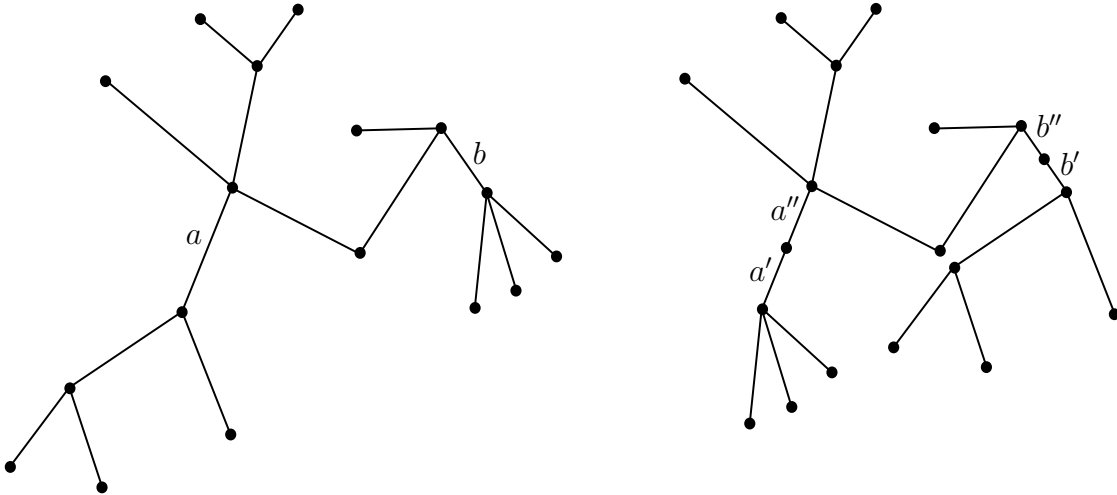


FIGURE 4. Double switch of a tree along the edge pair $\{a, b\}$.

²For a graph G and its edge e we denote by $G - e$ the subgraph of G obtained by deleting the edge e .

Characterizing realizable paired trees. In [Li04, Theorem 1] a sufficient and necessary condition was obtained: a paired tree is realizable if and only if after performing double switches successively along all of its pairs of edges the linking graph of the resulting tree is bipartite. This characterization of paired trees is analogous to the characterization of words being Gauss codes in [FO99, Theorem 6], where successive D-switches of linking graphs were used. It requires performing a sequence of successive operations (namely the double switches on the paired trees) in order to make use of it. From this viewpoint our main result in the present paper is the 2-dimensional analogue of [FO99, Theorem 10] (see also [RR78]) and we can say that we utilize the results of [Li04] just as Theorem 10 made practical use of Theorem 6 in [FO99].

In the present paper, we give a characterization of realizable paired trees in terms of the interlacement graphs of the trees. The benefit of this characterization is that it gives an equivalent condition for realizability in terms of an explicit property of the given paired tree. For instance, from our Theorem 2.1 immediately follows that if among the given system of circle pairs in S^2 there is a circle pair C_1, C_2 such that an *odd number* of other circle pairs are positioned with C_1, C_2 as in Figure 5, then this system is not realizable, for more details, see Corollary 2.2 and Corollary 2.4. (For an example of a paired tree and applying this criterion, see Figure 6.)

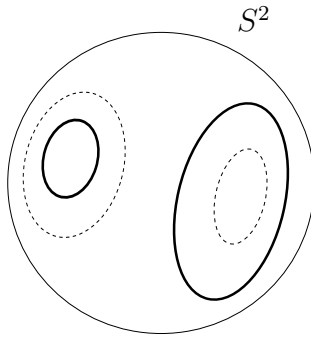


FIGURE 5. Two pairs of circles in S^2 . The two dashed circles form a pair and the two bold circles form a pair.

One of our most important observations is that the interlacement graphs of paired trees before and after a double switch are related to each other by a version of the local complementation for directed graphs, we explain this fact in detail during the proof of Proposition 4.1. (Although this seems to be an analogue of the local complementation in [FO99, Section 2], it is less obvious. Also note that local complementation can become useful for us only because we have found the right notion of directed interlacement

graphs.) By this knowledge we can compare the interlacement graphs and we can find properties of the graphs which remain unchanged during double switches. We formalize these properties in an algebraic way. The difficulty at this point is that the incidence matrix of a directed graph is not necessarily symmetric and hence we need a more sophisticated argument than the straightforward algebraization of [FO99, Section 4]. These and [Li04] together yield our main Theorem 2.1, which characterizes the realizable circle pairs in S^2 .

The paper is organized as follows. In Section 2 we announce our main results. In Section 3, we state Theorem 3.2, which is of central importance and allows us to prove our main results (its proof is deferred to later sections). In Section 4 we describe in a very detailed way how the double switch operation on paired trees acts on their interlacement graphs. By using this theory we prove Theorem 3.2 in Section 4 with the help of some quite technical statements. Finally, these technical statements are fully proved only in Section 5.

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2. MAIN RESULTS

So let $k \geq 1$ be fixed and $C \subset S^2$, $C = C_1 \cup \dots \cup C_{2k}$ be a disjoint union of $2k$ embedded circles arranged into the pairs $\mathbf{c}_j = \{C_j, C_{j+k}\}$, $1 \leq j \leq k$.

For any two circle pairs \mathbf{c} and \mathbf{d} let $[\mathbf{c}, \mathbf{d}]$ denote the set of circle pairs \mathbf{x} such that $\mathbf{c} \subseteq \mathbf{x}$ and $\mathbf{x} \subseteq \mathbf{d}$. Our main result is the following. As before, let $\{\mathbf{c}_1, \dots, \mathbf{c}_k\}$ be a given set of disjoint circle pairs in S^2 .

Theorem 2.1 (Main theorem). *The realizability of C is equivalent to the following condition: the set $\{1, \dots, k\}$ has a bipartition such that for any $i, j \in \{1, \dots, k\}$*

- (i) *i and j belong to the same class of the bipartition and*
- (ii) *$\mathbf{c}_i \subseteq \mathbf{c}_j$*

if and only if $[\mathbf{c}_i, \mathbf{c}_j]$ has an odd number of elements.

Theorem 2.1 and Theorem 2.3 below are analogous to Rosenstiehl's characterization of words being Gauss codes of immersed plane curves, see, for example, [FO99, Section 4, Theorem 10].

By taking $i = j$ in Theorem 2.1, we get

Corollary 2.2. *If C is realizable, then for any circle pair \mathbf{c} the number of circle pairs \mathbf{x} such that $\mathbf{c} \in \mathbf{x}$ and $\mathbf{x} \in \mathbf{c}$ both hold is even.*

In the following, we rephrase Theorem 2.1 in terms of directed graphs. For a vertex v of G let $N_{out}(v)$ and $N_{in}(v)$ denote the set of out-neighbors and the set of in-neighbors of v , respectively. The following theorem is clearly equivalent to Theorem 2.1.

Theorem 2.3. *C is realizable if and only if the set of vertices of G has a bipartition such that for any two vertices u, v of G the number of vertices in $N_{out}(u) \cap N_{in}(v)$ is odd if and only if*

- (i) *u and v belong to the same class of the partition and*
- (ii) *$v \in N_{out}(u)$.*

Proof. The graph G just expresses the relation \in so the statement follows immediately from Theorem 2.1. \square

By taking $i = j$ in Theorem 2.3, we get

Corollary 2.4. *In particular, if C is realizable, then the number of neighbors of any vertex v of the interlacement graph G which are connected by two oppositely directed edges to v is even.*

For example, the paired tree in Figure 6 is not realizable.

3. PROOF OF MAIN THEOREM

Definition 3.1 (Incidence matrix). A directed graph G with vertices p_1, \dots, p_m determines an $m \times m$ incidence matrix over \mathbb{Z}_2 as usual: let $L \in \mathbb{Z}_2^{m \times m}$ be the matrix such that the element $[L]_{i,j}$ in the i -th row and j -th column of L is

- equal to 1 if there is a directed edge of G going from p_j to p_i and
- equal to 0 in any other case.

We have the following very important result about the incidence matrix of the interlacement graph of a paired tree. For a vector space \mathbb{Z}_2^r let $\langle \cdot, \cdot \rangle: \mathbb{Z}_2^r \times \mathbb{Z}_2^r \rightarrow \mathbb{Z}_2$ denote the standard scalar product.

Theorem 3.2. *Let $L^{k \times k}$ be the incidence matrix of the interlacement graph G_0 of a paired tree T_0 with edge pairs $\{a_1, b_1\}, \dots, \{a_k, b_k\}$. Performing double switches along all the edge pairs $\{a_1, b_1\}, \dots, \{a_k, b_k\}$ successively gives the paired trees T_1, \dots, T_k and*

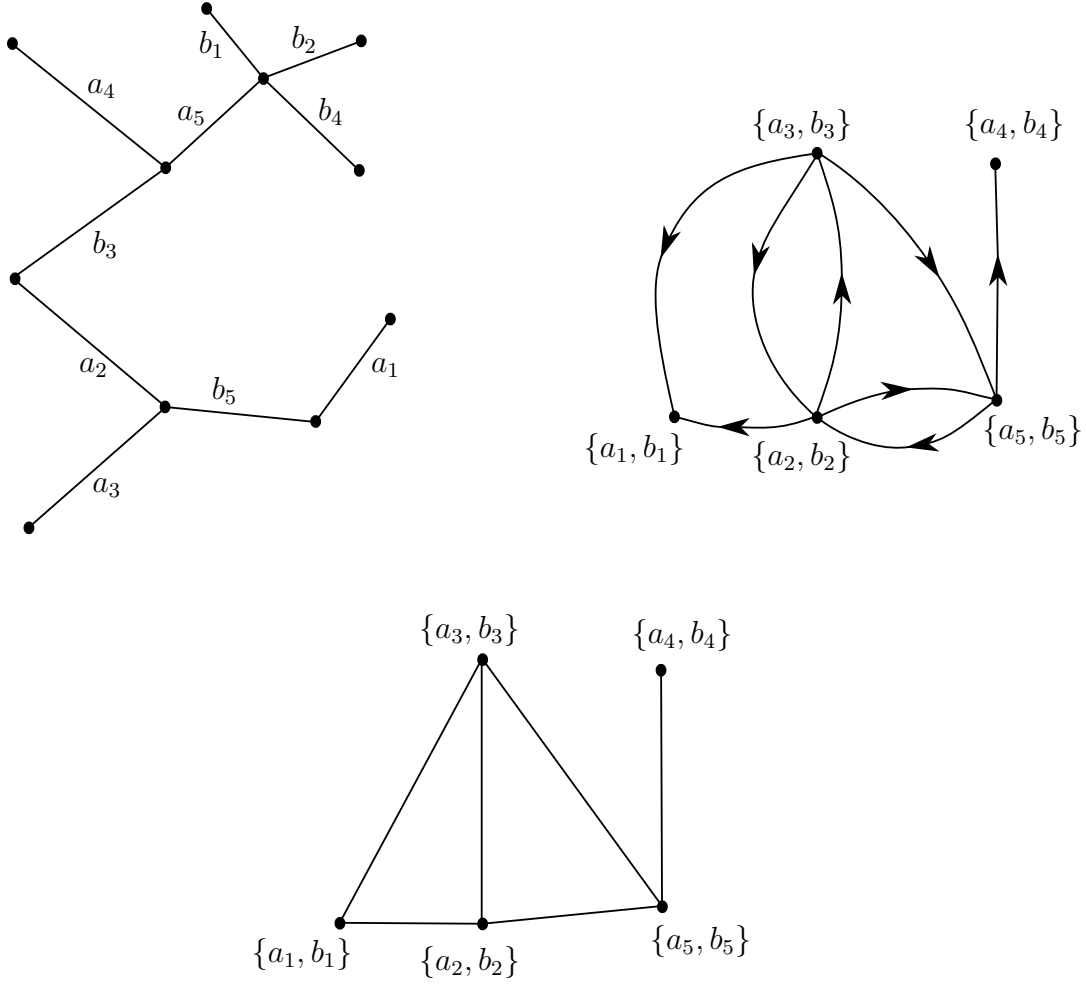


FIGURE 6. A paired tree, its interlacement graph on the right hand side and finally its linking graph. The vertices of the interlacement graph and linking graph correspond to the edge pairs of the tree. By Corollary 2.4 this paired tree is not realizable as we can see immediately from its interlacement graph.

the corresponding interlacement graphs G_1, \dots, G_k . Then the interlacement graph G_k of the paired tree T_k is bipartite if and only if there exists a vector $A \in \mathbb{Z}_2^k$ such that

$$\langle Lu, L^T v \rangle = \langle Lu, v \rangle (1 + \langle A, u + v \rangle)$$

holds for any standard basis vectors $u, v \in \mathbb{Z}_2^k$.

The proof of this theorem will be presented later in Section 4 on page 17. Nevertheless by using this result and [Li04, Theorem 1], now we can prove our main theorem:

Proof of Theorem 2.1. By [Li04, Theorem 1] it is easy to see that the interlacement graph G_k is bipartite if and only if the tree T_0 is realizable. On the other hand notice that for a basis vector u representing a vertex p the vector Lu shows for which vertices q there are oriented edges from p to q . Similarly $L^T u$ shows the oriented edges into p . So $\langle Lu, L^T v \rangle$ counts the vertices which are out-neighbors of u and in-neighbors of v . Also notice that the vector A accounts for the bipartition. Hence the algebraic condition in the statement of Theorem 3.2 is clearly equivalent to the condition about the circle pairs in Theorem 2.1. (Or equivalently we could say the same thing about Theorem 2.3.) \square

4. THE EFFECT OF THE DOUBLE SWITCH OPERATION AND LOCAL COMPLEMENTATION

In this section, we prove Theorem 3.2 but at first we describe in detail how double switch works.

If T_0 is a paired tree with edge pairs $\{\{a_i, b_i\} : 1 \leq i \leq k\}$, then its edges $a_2, b_2, \dots, a_k, b_k$ are naturally identified with edges of the paired tree T_1 obtained by double switch along $\{a_1, b_1\}$ as it was explained in the Introduction. We could identify $\{a_1, b_1\}$ with $\{a'_1, b'_1\}$ or $\{a''_1, b''_1\}$ in T_1 , let us fix the convention that $\{a_1, b_1\}$ is identified with $\{a'_1, b'_1\}$. Clearly, the number of vertices of the interlacement graph G_1 of T_1 is one more than that of the interlacement graph G_0 of T_0 and the vertices p_1, \dots, p_k of G_0 are identified with the corresponding vertices of G_1 . We denote the additional vertex of G_1 by q_1 , it corresponds to the edge pair $\{a''_1, b''_1\}$ and it has exactly the same in- and out-neighbors as p_1 . Similarly double switching successively along the next pairs of edges introduces the new vertices q_2, \dots, q_k into the new interlacement graphs G_2, \dots, G_k . The whole process also gives the corresponding new incidence matrices L_1, \dots, L_k .

Since each of the k double switches along the edge pairs increases the number of vertices of the interlacement graph and hence the size of the incidence matrix by one, it is convenient to put all the matrix $L_i \in \mathbb{Z}_2^{(k+i) \times (k+i)}$ into the upper left corner of a larger $2k \times 2k$ matrix denoted by $M_i \in \mathbb{Z}_2^{2k \times 2k}$ and declare the other elements of this larger M_i to be zero. The diagonal elements of M_i are all zero since the interlacement graph G_i has no loops.

To prove Theorem 3.2, we will need some more technical results. Starting with a paired tree T_0 , its linking graph G_0 with vertices p_1, \dots, p_k and the sequence of double

switch operations along the pairs of edges $\{a_1, b_1\}, \dots, \{a_k, b_k\}$ can be seen in a “vector space language” as follows.

The vector space \mathbb{Z}_2^{2k} splits as $\mathbb{Z}_2^{2k} = \mathbb{Z}_2^k \oplus \mathbb{Z}_2^k$ and from now on we identify the standard basis of the first \mathbb{Z}_2^k summand with the vertices p_1, \dots, p_k of G_0 . Likewise the second \mathbb{Z}_2^k summand is preserved for the new vertices q_1, \dots, q_k coming from the double switches later. So after double switching along $\{a_1, b_1\}$ the new vertex q_1 enters into the picture, the edges of the new interlacement graph G_1 are perhaps also different from that of G_0 , and the new incidence matrix $M_1 \in \mathbb{Z}_2^{2k \times 2k}$ of G_1 will have perhaps one more non-zero row and column: the $(k+1)$ th row and column represents how the new vertex q_1 participates in G_1 .

This implies that after $0 \leq i$ successive double switches along p_1, \dots, p_i , where $i \leq k$, the upper left $(k+i) \times (k+i)$ submatrix of M_i tells us how the edges of the interlacement graph G_i connect the vertices $p_1, \dots, p_k, q_1, \dots, q_i$. Therefore if u is one of the first $k+i$ standard basis elements of \mathbb{Z}_2^{2k} , then the vector $M_i u$ represents the out-neighbors of the vertex (represented by) u in G_i .

The following statement describes precisely how the incidence matrix changes under double switch of a tree.

Proposition 4.1. *Let $k \geq 1$ and $0 \leq i \leq k-1$. Let T_i be a paired tree with $2(k+i)$ edges, G_i be its interlacement graph with $k+i$ vertices and $M_i \in \mathbb{Z}_2^{2k \times 2k}$ be its incidence matrix as explained above. The paired tree obtained after double switching T_i along the pair of edges $\{a_{i+1}, b_{i+1}\}$ is denoted by T_{i+1} , its interlacement graph is denoted by G_{i+1} . If p_{i+1} denotes the vertex of G_i and G_{i+1} corresponding to $\{a_{i+1}, b_{i+1}\}$ and q_{i+1} denotes the new vertex of G_{i+1} , then*

- (1) *the incidence matrix $M_{i+1} \in \mathbb{Z}_2^{2k \times 2k}$ of T_{i+1} is obtained from M_i by*
 - $M_{i+1}u = M_i u + \langle M_i u, p_{i+1} \rangle ((M_i p_{i+1} + q_{i+1}) + \langle u, M_i p_{i+1} \rangle u)$, *if u is a standard basis element representing a vertex of $G_{i+1} - q_{i+1}$ and*
 - $M_{i+1}q_{i+1} = M_i p_{i+1}$,
- (2) *the incidence matrix M_i is obtained from M_{i+1} by*
 - $M_i u = M_{i+1} u + \langle M_{i+1} u, p_{i+1} \rangle ((M_{i+1} p_{i+1} + q_{i+1}) + \langle u, M_{i+1} p_{i+1} \rangle u)$, *if u is a standard basis element representing a vertex of G_i and*
 - $M_i q_{i+1} = 0$.

Remark 4.2. The same formulas hold for the matrices M_i^T instead of M_i .

Now we prove Proposition 4.1.

Proof of Proposition 4.1. We study how the interlacement graph G_i is changing during the double switch of T_i along $\{a_{i+1}, b_{i+1}\}$. We show that G_{i+1} is obtained from G_i by directed local complementation at p_{i+1} , where we add the vertex q_{i+1} with the same neighborhood as that of p_{i+1} (see in the Introduction and cf. [FO99, Section 2.2]). More precisely, denote by $NG_i(u)$ the subgraph of G_i whose vertices are connected (or identical) to u . We show that

- (i) $NG_{i+1}(p_{i+1}) = NG_i(p_{i+1})$,
- (ii) $NG_{i+1}(q_{i+1}) = NG_i(p_{i+1})$,
- (iii) for any vertex u of G_i we have $NG_{i+1}(u) = NG_i(u)$ if u is not connected by any edge to p_{i+1} in G_i and
- (iv) for any two vertices u, v in G_{i+1} not equal to p_{i+1} or q_{i+1} if u has an edge going into p_{i+1} and p_{i+1} has an edge going into v , then u has an edge going into v in G_{i+1} if and only if it has no such edge in G_i .

Note that the conditions (1) and (2) in the statement of Proposition 4.1 are clearly equivalent to (i)-(iv) if M_i denote the corresponding incidence matrices so it is enough to show that (i), (ii), (iii) and (iv) hold.

If x, y are two edges of a tree, then let \overline{xy} denote the path connecting (and containing) x and y . For two subgraphs G, H of a graph, let $G - H$ denote the subgraph of G which has the same vertices as G and has an edge if and only if G has it but H does not have it. We also introduce some other notations: if both of $\{a_j, b_j\} \sqsubset \{a_i, b_i\}$ and $\{a_j, b_j\} \sqsupset \{a_i, b_i\}$ hold, then we will write $\{a_j, b_j\} \parallel \{a_i, b_i\}$. If none of $\{a_j, b_j\} \sqsubset \{a_i, b_i\}$ and $\{a_j, b_j\} \sqsupset \{a_i, b_i\}$ hold, then we will write $\{a_j, b_j\} \sqsubset\sqsubset \{a_i, b_i\}$.

At first let us show that (i) holds. Take any edge pair $\{x, y\}$ of T_i different from $\{a_{i+1}, b_{i+1}\}$. Suppose $\{x, y\} \sqsubset \{a_{i+1}, b_{i+1}\}$. This means that exactly one of x, y is in $\overline{a_{i+1}b_{i+1}}$. Clearly this property still holds after double switch along $\{a_{i+1}, b_{i+1}\}$. Now suppose $\{x, y\} \sqsupset \{a_{i+1}, b_{i+1}\}$. This means that exactly one of a_{i+1} and b_{i+1} is in \overline{xy} , say a_{i+1} . Then after double switch, only b_{i+1} will be in \overline{xy} . So if a vertex of G_i is connected to p_{i+1} , then it will be connected to p_{i+1} in G_{i+1} in the same way. It follows that if $\{x, y\} \sqsubset\sqsubset \{a_{i+1}, b_{i+1}\}$ in T_i , then $\{x, y\} \sqsubset\sqsubset \{a_{i+1}, b_{i+1}\}$ in T_{i+1} as well because double switching twice along an edge pair keeps the paired tree (except the new dividing vertices on the edges along which we double switch) and so the interlacement relation. This argument shows that (i) holds.

Now it is easy to see that (ii) holds because (i) holds and the new vertex q_{i+1} comes from dividing the edges a_{i+1} and b_{i+1} as we explained in the Introduction.

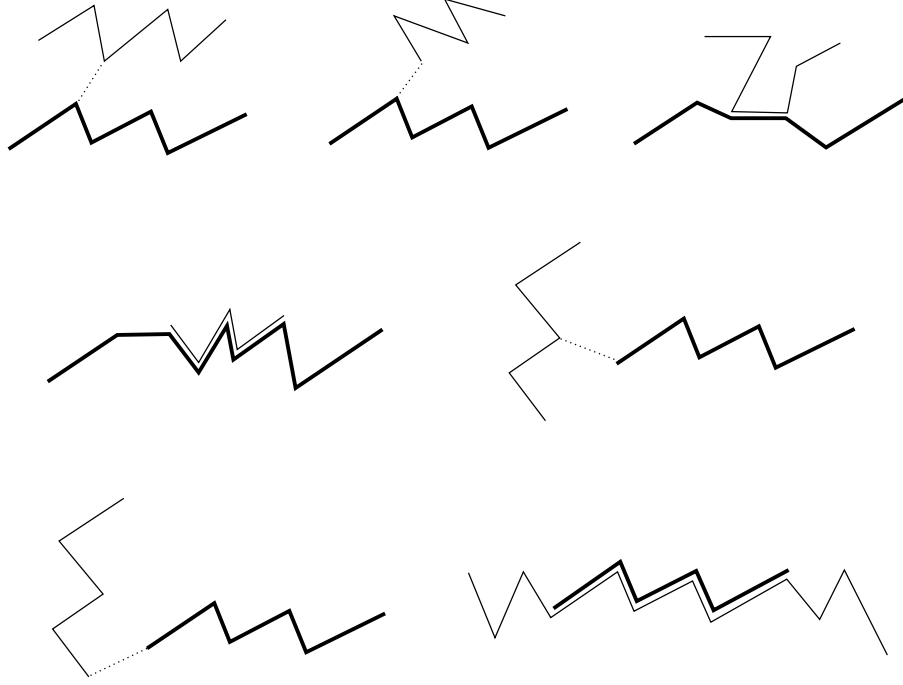


FIGURE 7. The bold zigzag symbolizes the path $\overline{a_{i+1}b_{i+1}}$ and the thin zigzag symbolizes the path \overline{xy} in the tree T_i . The dotted lines symbolize any paths (or just one vertex). In the first six cases \overline{xy} lies in one of the components of $T_i - a_{i+1} - b_{i+1}$.

To show (iii), take an edge pair $\{x, y\}$ different from $\{a_{i+1}, b_{i+1}\}$ such that $\{x, y\} \sqsubset \sqsubset \{a_{i+1}, b_{i+1}\}$ in T_i . There are two possibilities: \overline{xy} lies completely in one of the components of $T_i - a_{i+1} - b_{i+1}$, or \overline{xy} contains both of a_{i+1}, b_{i+1} , see Figure 7.

At first suppose that \overline{xy} lies completely in one of the components of $T_i - a_{i+1} - b_{i+1}$. Then clearly for any edge pair $\{u, v\}$ such that $\{u, v\} \sqsubset \{x, y\}$ in T_i we have the same in T_{i+1} as well. On the other hand, if $\{u, v\} \sqsupset \{x, y\}$, then assume \overline{uv} contains x and \overline{uv} does not contain y . It is easy to see that in the first three cases and in the fifth and sixth cases of Figure 7 the same holds after double switch and in the fourth case of Figure 7 after double switch still \overline{uv} contains one of x or y . If we suppose that \overline{xy} contains both of a_{i+1}, b_{i+1} (this is the seventh case of Figure 7), then the argument is similar. On the other hand, if $\{x, y\} \sqsupset \sqsupset \{u, v\}$ in T_i , then $\{x, y\} \sqsupset \sqsupset \{u, v\}$ in T_{i+1} as well since one more double switch along the same edge pair gives back T_i (up to dividing the edges a_{i+1}, b_{i+1}). So we proved (iii).

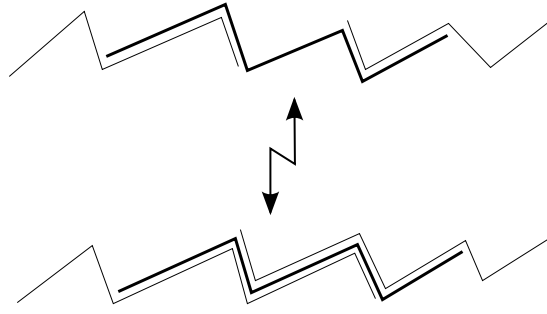


FIGURE 8. The path $\overline{a_{i+1}b_{i+1}}$ is symbolized by the bold zigzag. The thin zigzags symbolize the paths \overline{xy} and \overline{uv} in the tree T_i .

Now let us show (iv). Let $\{u, v\}$ and $\{x, y\}$ be different edge pairs in T_i , also different from $\{a_{i+1}, b_{i+1}\}$. There are some cases depending on how they are related to $\{a_{i+1}, b_{i+1}\}$ and to each other.

- (1) If in T_i the relations $\{u, v\} \sqsubset \{a_{i+1}, b_{i+1}\}$ and $\{x, y\} \sqsubset \{a_{i+1}, b_{i+1}\}$ both hold but not $\{u, v\} \sqsupset \{a_{i+1}, b_{i+1}\}$ and not $\{x, y\} \sqsupset \{a_{i+1}, b_{i+1}\}$, then $\{u, v\}$ and $\{x, y\}$ lie in the component of $T_i - a_{i+1} - b_{i+1}$ containing the inner vertices of $\overline{a_{i+1}b_{i+1}}$. In this case clearly $\{u, v\}$ and $\{x, y\}$ are related in T_{i+1} as they are in T_i .
- (2) If $\{u, v\} \sqsupset \{a_{i+1}, b_{i+1}\}$ and $\{x, y\} \sqsupset \{a_{i+1}, b_{i+1}\}$ but not $\{u, v\} \sqsubset \{a_{i+1}, b_{i+1}\}$ and not $\{x, y\} \sqsubset \{a_{i+1}, b_{i+1}\}$ in T_i , then assume u and x are in the component of $T_i - a_{i+1} - b_{i+1}$ containing the inner vertices of $\overline{a_{i+1}b_{i+1}}$. Then v and y are in the other components of $T_i - a_{i+1} - b_{i+1}$, which will be interchanged during double switch. Of course u and x are not in $\overline{a_{i+1}b_{i+1}}$. These imply that $\{u, v\}$ and $\{x, y\}$ are related in T_{i+1} the same way as in T_i .
- (3) If $\{u, v\} \parallel \{a_{i+1}, b_{i+1}\}$ and $\{x, y\} \parallel \{a_{i+1}, b_{i+1}\}$ in T_i , then $\{u, v\} \parallel \{x, y\}$ or $\{u, v\} \sqsubset \{x, y\}$ must hold in T_i , see Figure 8. In this case the relation between $\{u, v\}$ and $\{x, y\}$ in the new paired tree T_{i+1} changes as we claim: $\{u, v\} \parallel \{x, y\}$ (resp. $\{u, v\} \sqsubset \{x, y\}$) in T_{i+1} if and only if $\{u, v\} \sqsubset \{x, y\}$ (resp. $\{u, v\} \parallel \{x, y\}$) in T_i . See Figure 8.
- (4) Or else, if $\{u, v\} \parallel \{a_{i+1}, b_{i+1}\}$ and $\{x, y\} \sqsubset \{a_{i+1}, b_{i+1}\}$ or $\{x, y\} \sqsupset \{a_{i+1}, b_{i+1}\}$ in T_i , then the possible configurations can be seen in Figure 9. It is easy to check that our claim holds, details are left to the reader.
- (5) Finally, if none of the above cases hold but $\{u, v\} \sqsupset \{a_{i+1}, b_{i+1}\}$ and $\{x, y\} \sqsubset \{a_{i+1}, b_{i+1}\}$, then see Figure 10.

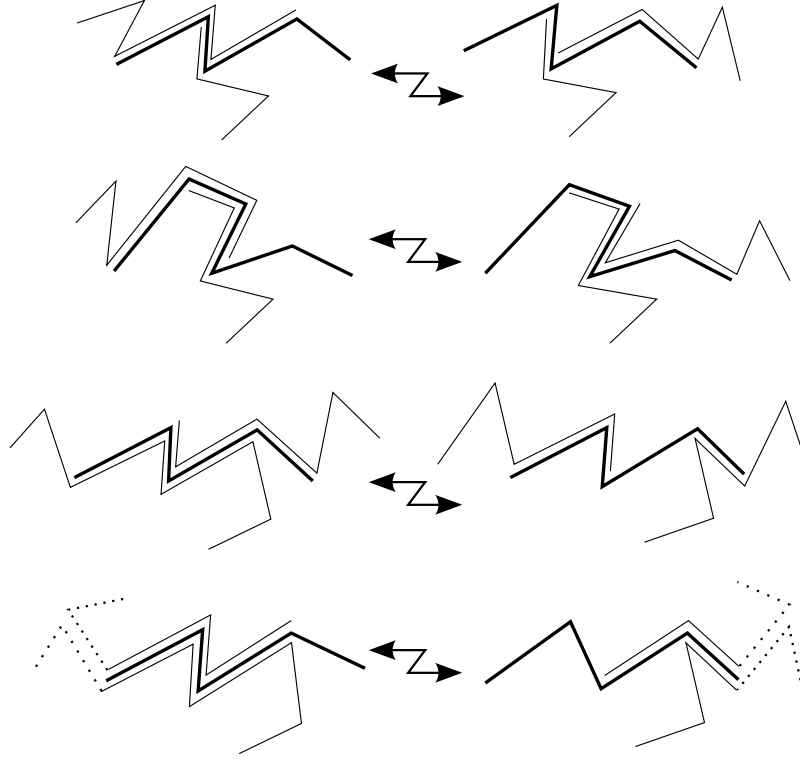


FIGURE 9. The bold zigzag symbolizes the path $\overline{a_{i+1}b_{i+1}}$ and the thin (possibly dotted) zigzags symbolize the paths \overline{xy} and \overline{uv} in the tree T_i . $\{u, v\} \parallel \{a_{i+1}, b_{i+1}\}$ holds in any cases, $\{x, y\} \sqsubset \{a_{i+1}, b_{i+1}\}$ holds in the first four and $\{x, y\} \sqsupset \{a_{i+1}, b_{i+1}\}$ holds in the last four cases. The relations between $\{u, v\}$ and $\{x, y\}$ in the first six cases are the following: $\{u, v\} \sqsubset \{x, y\}$, $\{u, v\} \sqsubset\sqsubset \{x, y\}$, $\{u, v\} \parallel \{x, y\}$, $\{u, v\} \sqsubset \{x, y\}$, $\{u, v\} \sqsubset \{x, y\}$, $\{u, v\} \sqsupset\sqsupset \{x, y\}$.

□

To prove Theorem 3.2 we will need the following important proposition. Let $\mathcal{B}(\mathbb{Z}_2^r)$ denote the set of standard basis elements of a vector space \mathbb{Z}_2^r .

Proposition 4.3. *Let $0 \leq i \leq k$ be fixed and let $A \in \mathbb{Z}_2^{2k}$ be a fixed vector. If for any $x, y \in \mathcal{B}(\mathbb{Z}_2^{2k})$ we have*

$$\langle M_i x, M_i^T y \rangle = \langle M_i x, y \rangle (1 + \langle A, x + y \rangle),$$

then there exist vectors $B, C \in \mathbb{Z}_2^{2k}$ such that we have for any $u, v \in \mathcal{B}(\mathbb{Z}_2^{2k})$

- (1) (i) *the $k + i + 2, \dots, 2k$ -th coordinates of B are all zero and*
- (ii) $\langle M_{i+1} u, M_{i+1}^T v \rangle = \langle M_{i+1} u, v \rangle (1 + \langle B, u + v \rangle)$

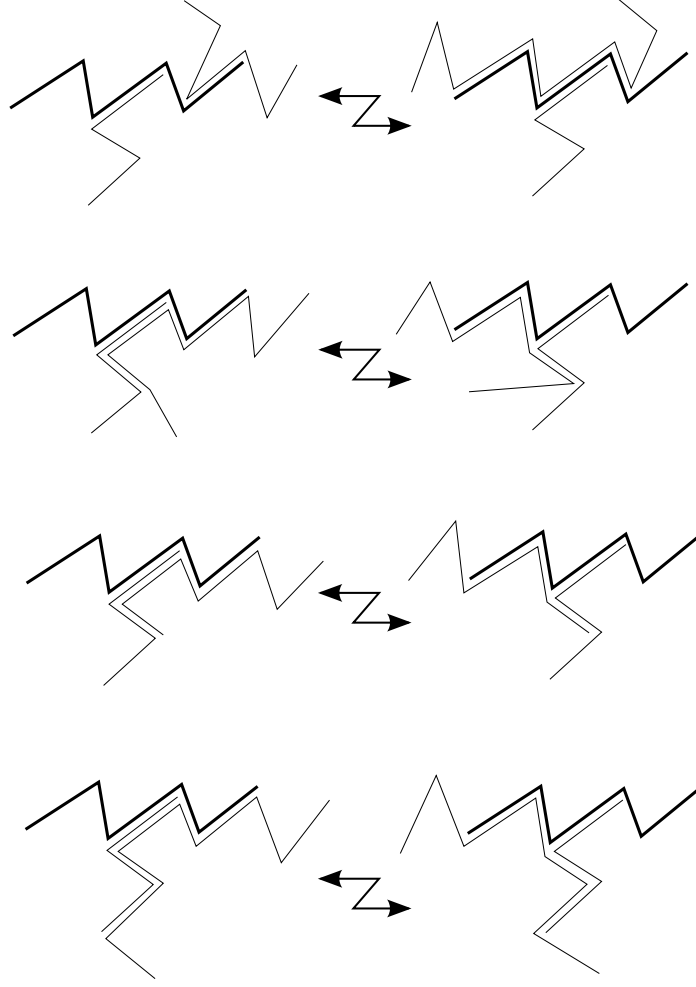


FIGURE 10. The bold zigzag symbolizes the path $\overline{a_{i+1}b_{i+1}}$ and the thin zigzags symbolize the paths \overline{xy} and \overline{uv} in the tree T_i . $\{u, v\} \supset \{a_{i+1}, b_{i+1}\}$ and $\{x, y\} \subset \{a_{i+1}, b_{i+1}\}$ hold in all cases.

(2) if $0 \leq i \leq k-1$ and the $k+i+1, \dots, 2k$ -th coordinates of A are all zero, and

$$\langle M_{i-1}u, M_{i-1}^T v \rangle = \langle M_{i-1}u, v \rangle (1 + \langle C, u + v \rangle)$$

if $1 \leq i \leq k$.

The proof of this proposition is quite technical and so it will be presented only in the next section. Now we prove Theorem 3.2.

Proof of Theorem 3.2. First suppose that there exists a vector $A \in \mathbb{Z}_2^k$ such that

$$\langle Lu, L^T v \rangle = \langle Lu, v \rangle (1 + \langle A, u + v \rangle)$$

holds for any $u, v \in \mathcal{B}(\mathbb{Z}_2^k)$. As explained at the beginning of Section 4, we consider L as a $2k \times 2k$ matrix M_0 and also the vector A as a vector in \mathbb{Z}_2^{2k} whose $k+1, \dots, 2k$ -th coordinates are all zero. Then still

$$\langle M_0 u, M_0^T v \rangle = \langle M_0 u, v \rangle (1 + \langle A, u + v \rangle)$$

holds for any $u, v \in \mathcal{B}(\mathbb{Z}_2^{2k})$. Then by applying Proposition 4.3 (1) iteratively, after performing the double switches the final incidence matrix M_k of the tree T_k satisfies the same type of equation with some vector $A_k \in \mathbb{Z}_2^{2k \times 2k}$ instead of A . Clearly A_k determines a bipartition of the interlacement graph of the tree T_k by the conditions $\langle A_k, u \rangle = 0$ or $\langle A_k, u \rangle = 1$ for a vertex represented by u .

Since in the interlacement graph of the tree T_k for every vertex the adjacent edges are duplicated due to the double switches, we have $\langle M_k u, M_k^T v \rangle = 0$ for any $u, v \in \mathcal{B}(\mathbb{Z}_2^{2k})$. Then the equation

$$0 = \langle M_k u, M_k^T v \rangle = \langle M_k u, v \rangle (1 + \langle A_k, u + v \rangle)$$

implies that if there is an edge going from u to v , then u and v are in different classes of the partition. So the interlacement graph of T_k is bipartite.

Now suppose that the interlacement graph of the tree T_k is bipartite. Then there exists a vector $A_k \in \mathbb{Z}_2^{2k}$ such that

$$\langle M_k u, v \rangle (1 + \langle A_k, u + v \rangle) = 0$$

holds for any $u, v \in \mathcal{B}(\mathbb{Z}_2^{2k})$. Since $\langle M_k u, M_k^T v \rangle = 0$ for any $u, v \in \mathcal{B}(\mathbb{Z}_2^{2k})$, the equation

$$\langle M_k u, M_k^T v \rangle = \langle M_k u, v \rangle (1 + \langle A_k, u + v \rangle)$$

holds for any $u, v \in \mathcal{B}(\mathbb{Z}_2^{2k})$. Hence by Proposition 4.3 (2), the same type of equation holds for the matrix M_0 and a vector $A_0 \in \mathbb{Z}_2^{2k}$ as well. Then the $k \times k$ upper left submatrix L of M_0 and the vector A obtained from A_0 by taking only the first k coordinates satisfy

$$\langle Lu, L^T v \rangle = \langle Lu, v \rangle (1 + \langle A, u + v \rangle)$$

for any $u, v \in \mathcal{B}(\mathbb{Z}_2^k)$. □

5. PROOF OF PROPOSITION 4.3

At first we prove two lemmas. Let $r \geq 1$ and let $\mathcal{B}(\mathbb{Z}_2^r)$ denote the set of standard basis elements of \mathbb{Z}_2^r . For a matrix $M \in \mathbb{Z}_2^{r \times r}$ let $S(M) \in \mathbb{Z}_2^{r \times r}$ be the matrix defined by $\langle S(M)x, y \rangle = \langle Mx, y \rangle \langle My, x \rangle$, where $x, y \in \mathcal{B}(\mathbb{Z}_2^r)$. Clearly we have $S(M) = S(M)^T = S(M^T)$.

Suppose the diagonal of M is fully zero. Let $p, p' \in \mathcal{B}(\mathbb{Z}_2^r)$ fixed such that $p \neq p'$. For any $u \in \mathcal{B}(\mathbb{Z}_2^r)$ such that $u \neq p'$ let $P(u) = \langle Mu, p \rangle ((Mp + p') + \langle u, Mp \rangle u)$ and $P(u)^T = \langle M^T u, p \rangle ((M^T p + p') + \langle u, M^T p \rangle u)$. (Compare these formulas with Proposition 4.1.) Also suppose that

$$(5.1) \quad \langle M^T v, p \rangle \langle Mu, p' \rangle + \langle Mu, p \rangle \langle M^T v, p' \rangle = 0$$

for all $u, v \in \mathcal{B}(\mathbb{Z}_2^r)$ and M is such that

$$(5.2) \quad \langle Mp, p' \rangle = \langle Mp', p \rangle = 0.$$

Lemma 5.1. *Let $A \in \mathbb{Z}_2^r$ be a fixed vector. If for any $x, y \in \mathcal{B}(\mathbb{Z}_2^r)$ we have*

$$\langle Mx, M^T y \rangle = \langle Mx, y \rangle (1 + \langle A, x + y \rangle),$$

then for any $u, v \in \mathcal{B}(\mathbb{Z}_2^r)$ such that $u, v \neq p'$, we have

$$(5.3) \quad \langle Mu + P(u), M^T v + P(v)^T \rangle = \langle Mu + P(u), v \rangle (1 + \langle A + S(M)p, u + v \rangle).$$

Proof. Let S denote $S(M)$ for short. The right hand side of (5.3) is equal to

$$\langle Mu, v \rangle (1 + \langle A, u + v \rangle) + \langle Mu, v \rangle \langle Sp, u + v \rangle + \langle P(u), v \rangle (1 + \langle A + Sp, u + v \rangle).$$

By subtracting $\langle Mu, M^T v \rangle = \langle Mu, v \rangle (1 + \langle A, u + v \rangle)$ from both sides of (5.3), it is enough to show that

$$(5.4) \quad \langle Mu, P^T(v) \rangle + \langle P(u), M^T v \rangle + \langle P(u), P(v)^T \rangle = \langle Mu, v \rangle \langle Sp, u + v \rangle + \langle P(u), v \rangle (1 + \langle A + Sp, u + v \rangle).$$

By the definition of $P(v)^T$, we have

$$\begin{aligned}\langle Mu, P(v)^T \rangle &= \langle Mu, \langle M^T v, p \rangle ((M^T p + p') + \langle v, M^T p \rangle v) \rangle = \\ &\quad \langle M^T v, p \rangle \langle Mu, M^T p + p' \rangle + \langle Sp, v \rangle \langle Mu, v \rangle\end{aligned}$$

and similarly

$$\langle P(u), M^T v \rangle = \langle Mu, p \rangle \langle Mp + p', M^T v \rangle + \langle Sp, u \rangle \langle u, M^T v \rangle.$$

Since by (5.1) $\langle M^T v, p \rangle \langle Mu, p' \rangle + \langle Mu, p \rangle \langle p', M^T v \rangle = 0$, we have

$$\begin{aligned}\langle Mu, P(v)^T \rangle + \langle P(u), M^T v \rangle &= \\ \langle M^T v, p \rangle \langle Mu, M^T p \rangle + \langle Mu, p \rangle \langle Mp, M^T v \rangle + \langle Sp, v \rangle \langle Mu, v \rangle + \langle Sp, u \rangle \langle u, M^T v \rangle &= \\ \langle M^T v, p \rangle \langle Mu, p \rangle (1 + \langle A, u + p \rangle) + \langle Mu, p \rangle \langle p, M^T v \rangle (1 + \langle A, p + v \rangle) + \\ \langle Sp, v \rangle \langle Mu, v \rangle + \langle Sp, u \rangle \langle u, M^T v \rangle &= \\ \langle Mu, p \rangle \langle M^T v, p \rangle \langle A, u + v \rangle + \langle Mu, v \rangle \langle Sp, u + v \rangle.\end{aligned}$$

Now we have two cases.

(1) Suppose that $u \neq v$. Then, we have

$$\langle P(u), v \rangle = \langle \langle Mu, p \rangle ((Mp + p') + \langle u, Mp \rangle u), v \rangle = \langle Mu, p \rangle \langle Mp, v \rangle,$$

since $v \neq p'$ and $u \neq v$. So after putting the above results into (5.4) and simplifying, we have to prove that

$$\begin{aligned}\langle Mu, v \rangle \langle Sp, u + v \rangle + \langle P(u), P(v)^T \rangle &= \\ \langle Mu, v \rangle \langle Sp, u + v \rangle + \langle Mu, p \rangle \langle Mp, v \rangle (1 + \langle Sp, u + v \rangle),\end{aligned}$$

which is equivalent to

$$\langle P(u), P(v)^T \rangle = \langle Mu, p \rangle \langle Mp, v \rangle (1 + \langle Sp, u + v \rangle).$$

Furthermore, we have

$$\begin{aligned} \langle P(u), P(v)^T \rangle &= \\ \langle \langle Mu, p \rangle ((Mp + p') + \langle u, Mp \rangle u), \langle M^T v, p \rangle ((M^T p + p') + \langle v, M^T p \rangle v) \rangle &= \\ \langle Mu, p \rangle \langle M^T v, p \rangle (\langle p', p' \rangle + \langle Mp, M^T p \rangle + \langle v, M^T p \rangle \langle Mp, v \rangle + \langle u, Mp \rangle \langle u, M^T p \rangle) &= \\ \langle Mu, p \rangle \langle M^T v, p \rangle + 0 + \langle Mu, p \rangle \langle M^T v, p \rangle (\langle M^T v, p \rangle \langle v, M^T p \rangle + \langle Mu, p \rangle \langle u, Mp \rangle), \end{aligned}$$

where we used (5.2) and that $\langle Mp, M^T p \rangle = 0$. But this is equal to

$$\langle Mu, p \rangle \langle M^T v, p \rangle (1 + \langle Sp, u \rangle + \langle Sp, v \rangle).$$

This finishes the proof for the case $u \neq v$.

(2) Now suppose that $u = v$.

Then

$$\langle P(u), v \rangle = \langle P(u), u \rangle = \langle \langle Mu, p \rangle ((Mp + p') + \langle u, Mp \rangle u), u \rangle = 2\langle Mu, p \rangle \langle Mp, u \rangle = 0.$$

Also

$$\langle Mu, P(v)^T \rangle + \langle P(u), M^T v \rangle = \langle Mu, p \rangle \langle M^T u, p \rangle \langle A, 2u \rangle + \langle Mu, u \rangle \langle Sp, 2u \rangle = 0.$$

So we have to prove that

$$\langle P(u), P(u)^T \rangle = 0.$$

For this, we have

$$\begin{aligned} \langle P(u), P(u)^T \rangle &= \\ \langle \langle Mu, p \rangle ((Mp + p') + \langle u, Mp \rangle u), \langle M^T u, p \rangle ((M^T p + p') + \langle u, M^T p \rangle u) \rangle &= \\ 2\langle Mu, p \rangle \langle Mp, u \rangle + 0 &= 0. \end{aligned}$$

Which again finishes the proof for the case $u = v \neq p'$.

□

Now we show the similar results for the cases when at least one of u, v is equal to p' . Again suppose M is such that

$$(5.5) \quad \langle Mp, p' \rangle = \langle Mp', p \rangle = 0.$$

The necessary statements are the following.

Lemma 5.2. *Let $A \in \mathbb{Z}_2^r$ be a fixed vector. If for any $x, y \in \mathcal{B}(\mathbb{Z}_2^r)$ we have*

$$\langle Mx, M^T y \rangle = \langle Mx, y \rangle (1 + \langle A, x + y \rangle),$$

then

(1) *we have for all $u \in \mathcal{B}(\mathbb{Z}_2^r)$, $u \neq p'$*

(a)

(5.6)

$$\langle Mu + P(u), M^T p \rangle = \langle Mu + P(u), p' \rangle (1 + \langle A + S(M)p + \langle A, p \rangle p', u + p' \rangle)$$

if $\langle Mu, p' \rangle = 0$ and $\langle A, p' \rangle = 0$, and

$$(5.7) \quad \langle Mu + P(u), 0 \rangle = \langle Mu + P(u), p' \rangle (1 + \langle A + S(M)p + \langle A, p \rangle p', u + p' \rangle)$$

if $\langle Mu, p' \rangle = \langle Mu, p \rangle$,

(b)

$$(5.8) \quad \langle Mp, M^T u + P^T(u) \rangle = \langle Mp, u \rangle (1 + \langle A + S(M)p + \langle A, p \rangle p', p' + u \rangle),$$

if $\langle A, p' \rangle = 0$, and clearly

$$(5.9) \quad \langle 0, M^T u + P^T(u) \rangle = \langle 0, u \rangle (1 + \langle A + S(M)p + \langle A, p \rangle p', p' + u \rangle),$$

and

(2) *we have*

$$(5.10) \quad \langle Mp, M^T p \rangle = \langle Mp, p' \rangle (1 + \langle A + S(M)p + \langle A, p \rangle p', p' + p' \rangle).$$

and obviously $\langle 0, 0 \rangle = \langle 0, p' \rangle (1 + \langle A + S(M)p + \langle A, p \rangle p', p' + p' \rangle)$.

Proof. Let S denote $S(M)$ for short. For (1)(a) let us try to show

$$\begin{aligned} \langle Mu + \langle Mu, p \rangle p' + \langle Mu, p \rangle Mp + \langle Sp, u \rangle u, M^T p \rangle = \\ \langle Mu + \langle Mu, p \rangle p' + \langle Mu, p \rangle Mp + \langle Sp, u \rangle u, p' \rangle (1 + \langle A + Sp + \langle A, p \rangle p', u + p' \rangle). \end{aligned}$$

The right hand side is equal to $\langle Mu, p + p' \rangle (1 + \langle A + Sp + \langle A, p \rangle p', u + p' \rangle)$ by (5.5). So (5.7) is obvious. To show (5.6) suppose $\langle Mu, p' \rangle = 0$ and $\langle A, p' \rangle = 0$. Then we have to prove

$$\langle Mu, M^T p \rangle + \langle Mu, p \rangle \langle Mp, M^T p \rangle + \langle Sp, u \rangle = \langle Mu, p \rangle (1 + \langle A + Sp + \langle A, p \rangle p', u + p' \rangle),$$

where we used (5.5) and $\langle Sp, u \rangle \langle u, M^T p \rangle = \langle Sp, u \rangle$. Since $\langle Mp, M^T p \rangle = 0$, this is further equivalent to

$$\langle Mu, p \rangle (1 + \langle A, u + p \rangle) + \langle Sp, u \rangle = \langle Mu, p \rangle (1 + \langle A + Sp + \langle A, p \rangle p', u + p' \rangle)$$

which holds if $\langle A, p \rangle = \langle A + \langle A, p \rangle p', p' \rangle$. Since $\langle A, p' \rangle = 0$, we get the statement.

For (1)(b) since (5.9) obviously holds, suppose $\langle A, p' \rangle = 0$. We have to show that

$$\begin{aligned} \langle Mp, M^T u + \langle M^T u, p \rangle p' + \langle M^T u, p \rangle M^T p + \langle Sp, u \rangle u \rangle = \\ \langle Mp, u \rangle (1 + \langle A + Sp + \langle A, p \rangle p', p' + u \rangle). \end{aligned}$$

This is equivalent to

$$\langle Mp, M^T u \rangle + \langle Sp, u \rangle = \langle Mp, u \rangle (1 + \langle A + Sp + \langle A, p \rangle p', p' + u \rangle)$$

which is further equivalent to

$$\langle Mp, u \rangle (1 + \langle A, p + u \rangle) + \langle Sp, u \rangle = \langle Mp, u \rangle (1 + \langle A + Sp + \langle A, p \rangle p', p' + u \rangle)$$

which holds similarly to the previous case.

Finally (2) holds because both sides are equal to zero. \square

Now we are ready to prove Proposition 4.3.

Proof of Proposition 4.3. Let $A \in \mathbb{Z}_2^{2k}$ be a fixed vector such that its $k+i+1, \dots, 2k$ -th coordinates are all zero. Assume that for any standard basis elements $x, y \in \mathbb{Z}_2^{2k}$ we

have

$$\langle M_i x, M_i^T y \rangle = \langle M_i x, y \rangle (1 + \langle A, x + y \rangle).$$

In order to prove (1) we want to show that there exists a vector $B \in \mathbb{Z}_2^{2k}$ such that we have for any standard basis elements $u, v \in \mathbb{Z}_2^{2k}$

$$\langle M_{i+1} u, M_{i+1}^T v \rangle = \langle M_{i+1} u, v \rangle (1 + \langle B, u + v \rangle).$$

For the reader's convenience we recall that M_i denotes the incidence matrix of the paired tree T_i and M_{i+1} is obtained from M_i according to the formulas of Proposition 4.1.

Note that with the roles $M = M_i$, $p = p_{i+1}$ and $p' = q_{i+1}$

$$\langle M^T v, p \rangle \langle M u, p' \rangle + \langle M u, p \rangle \langle M^T v, p' \rangle = 0$$

holds because $\langle M u, p' \rangle = \langle M^T v, p' \rangle = 0$ for every $u, v \in \mathcal{B}(\mathbb{Z}_2^{2k})$ since in M the row and column corresponding to p' are entirely zero. Let

$$B = A + S(M_i)p_{i+1} + \langle A, p_{i+1} \rangle q_{i+1}.$$

If $u, v \neq q_{i+1}$, then

$$\langle B, u + v \rangle = \langle A + S(M_i)p_{i+1}, u + v \rangle$$

and by using Lemma 5.1 we get our claim (notice that (5.2) holds). And since (5.5) holds and also $\langle A, p' \rangle = 0$, if at least one of u, v is equal to q_{i+1} , then by (5.6), (5.8) in (1) of Lemma 5.2 and by (5.10) in (2) of Lemma 5.2 we get our claim. Notice that the $k + i + 2, \dots, 2k$ -th coordinates of B are all zero.

To prove (2), we proceed similarly. Take $M = M_i$, $p = p_i$ and $p' = q_i$. In this case

$$\langle M^T v, p \rangle \langle M u, p' \rangle + \langle M u, p \rangle \langle M^T v, p' \rangle = 0$$

holds because $\langle M u, p' \rangle = \langle M u, p \rangle$ and $\langle M^T v, p' \rangle = \langle M^T v, p \rangle$ for every $u, v \in \mathcal{B}(\mathbb{Z}_2^{2k})$ since the rows and columns in M , respectively, corresponding to p and p' are the same.

Let

$$C = A + S(M_i)p_i + \langle A, p_i \rangle q_i.$$

If $u, v \neq q_i$, then

$$\langle C, u + v \rangle = \langle A + S(M_i)p_i, u + v \rangle$$

and by using Lemma 5.1 again we get our claim. And if at least one of u, v is equal to q_i , then by (5.7), (5.9) and (2) of Lemma 5.2 we get our claim. \square

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